Section 14.2

Derivatives of vector-valued functions

For a function $f : \mathbb{R} \to \mathbb{R}$, the derivative f' is defined as

We generalize the definition to vector-valued functions $\vec{r} : \mathbb{R} \to \mathbb{R}^n$.

Definition: Derivative and Tangent Vector
Let $\vec{r}(t) = f(t)\vec{\imath} + g(t)\vec{\jmath} + h(t)\vec{k}$, where f, g, h are differentiable functions on (a, b). Then $\vec{r}$ has a **derivative** (is **differentiable**) on (a, b) and

$$\vec{r}\,'(t) = f'(t)\vec{\imath} + g'(t)\vec{\jmath} + h'(t)\vec{k}$$

If $\vec{r}\,'(t) \neq \vec{0}$, then $\vec{r}\,'(t)$ is a **tangent vector** at the point corresponding to $\vec{r}(t)$.

Example 14.2.1. Find the derivative of $\vec{r}(t) = \langle t^2, 3t^2, \frac{t^3}{6} \rangle$

If $\vec{r}\,'(t_0) = \vec{0}$, then $\vec{r}(t_0)$ has no tangent vector at the point $\vec{r}(t_0)$. The curve $\vec{r}(t)$ may have a cusp at $r(t_0)$. If $\vec{r}\,'(t) \neq \vec{0}$ and f', g', t' exist for all t, we say $\vec{r}$ is a **smooth** function.

Recall that we can normalize any non-zero vector, so too can tangent vectors be normalized.

Definition: Unit Tangent Vectors
Let $\vec{r}(t) = f(t)\vec{\imath} + g(t)\vec{\jmath} + h(t)\vec{k}$ be a smooth curve. The **unit tangent vector** at of $\vec{r}$ at t is

$$\vec{T} =$$

Derivative rules

Because the derivative of a vector-valued function is defined via component wise differentiation, we have the following derivative rules. Here $\vec{u}(t), \vec{v}(t)$ are our differentiable vector functions, $f(t)$ a real valued differentiable function.

1.

2.

3.

4.

5.

6.

Example 14.2.2. Let $\vec{u}(t) = t\vec{\imath} + t^2\vec{\jmath} - \sin t\vec{k}$. Compute $\dfrac{d}{dt}(t^2\vec{u}(t))$.

Higher order derivatives and antiderivatives work exactly as you expect, component wise repeated differentiation or in the latter case the reverse operation.

If $\vec{R}(t) = F(t)\vec{\imath} + G(t)\vec{\jmath} + H(t)\vec{k}$ is such that $\vec{R}' = \vec{r}$ then

Definition: Indefinite Integral of Vector-Valued Functions
For a function $\vec{r}(t) = f(t)\vec{\imath} + g(t)\vec{\jmath} + h(t)\vec{k}$, the indefinite integral $\int \vec{r}\,dt$ is the set of all antiderivatives of $\vec{r}$. If $\vec{R}(t)$ is one such antiderivative the

$$\int \vec{r}(t)dt = \vec{R}(t) + \vec{C}$$

where $\vec{C}$ is an arbitrary constant vector.

Like the derivative, we define the definite integral of the vector function $\vec{r}(t) = f(t)\vec{\imath} + g(t)\vec{\jmath} + h(t)\vec{k}$ in the obvious way. As a lovely consequence, we get to reuse the Fundamental Theorem of Calculus from Cal I for evaluation!